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General Rational Approximants in N-Variables

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General off-diagonal rational approximants are defined from power series in N-variables. These approximants are generalizations of Chisholm approximants, which have been defined recently. A systematic method of ordering the defining equations is given, and this ordering is used in the prong method for solving the defining equations. Conditions for nondegeneracy of the defining equations are found in terms of the coefficients in the one-variable series obtained from the N-variable series by equating all but one variable to zero. A number of properties of the approximants are found; the two variable approximants being considered in more detail.

1. INTRODUCTION

In a recent paper, Chisholm [1], a method of defining "diagonal" rational approximants to a function of two variables defined by its power series was introduced. In another paper, Chisholm and McEwan [2], this idea was extended to cover functions of N-variables. These Chisholm approximants (CA's) may be referred to as diagonal approximants, as the maximum powers of all variables, in both the numerator and denominator, are equal to some integer m.

In this paper, these ideas will be generalized to off-diagonal approximants in the sense that the maximum powers of each variable may be different in the numerator and in the denominator, i.e., rational approximants with maximum powers

$$m_i, \qquad i = 1,..., N, \text{ in the numerator,}$$

and

$$n_i, \qquad i = 1,..., N, \text{ in the denominator,}$$

will be defined. One may thus consider the approximants in terms of hyper-rectangular boxes in the lattice space of indices, instead of hypercubes as for Chisholm approximants. Of course, these approximants will include CA's as special cases.

The coefficients in the rational approximants will, as for CA's, be defined as the solution of a set of linear equations, called the defining equations. The method of definition of these defining equations, involving the idea of "prongs," will yield a number of useful results concerning the conditions necessary for the set of equations to be nondegenerate. It will be shown that these conditions are closely linked with the conditions for the one variable power series, obtained from the N-variable power series by equating all but one of the variables to zero, to be normal series in the sense of Padé approximants.

A systematic method, called the prong method, of solving the defining equations will be constructed. This method may of course be used for solving the defining equations for the special case of CA's. The prong method introduced here is a generalization of the method used in a previous paper, Hughes Jones and Makinson [5], in which Chisholm approximants for two variables were discussed.

A simple-off-diagonal or symmetric-off-diagonal (SOD) approximant will be one in which the maximum powers m_i, in the numerator are all equal, to m say, and the maximum powers, n_i, in the denominator are all equal, to n say. Such approximants are probably of use in dealing with functions symmetric in the variables. SOD approximants to the Beta function, a symmetric function in two variables, have been calculated (Graves-Morris, Hughes Jones and Makinson [4]). The results are encouraging.

A general-off-diagonal (GOD) approximant, in this paper, will be one in which all the maximum powers may be different. In Section 2, the general-off-diagonal approximants are defined and the defining equations are grouped together in a systematic way. In Section 3, two variable approximants are studied in some detail and in Section 4, the methods and theorems of Section 3 are extended to cover N-variable approximants.

Apart from the theorems about the conditions for nondegeneracy of the approximants, the following properties are found:

(i) Reduction to the corresponding Padé approximant when all but one of the variables are equated to zero.

(ii) Reduction to an approximant in a smaller number of variables when some of the variables are equated to zero, i.e., the approximants satisfy a projection property.

(iii) Factorization of the approximant into the product of two approximants when the original function is the product of two functions involving independent sets of variables.

(iv) The approximant formed from the reciprocal series is the reciprocal of the approximant formed from the original series.

It is not difficult to construct more general schemes to define more general approximants than those considered in this paper. It is, however, not yet clear whether any other schemes will yield a systematic method of solution or whether other approximants will have all the properties (i)–(iv).

2. The Definition of Rational Approximants

Let $f(z_1, ..., z_N)$ be a function of N-variables defined by a possibly formal power series expansion

$$f(z_1, ..., z_N) = \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_N=0}^{\infty} c_{\alpha_1 \cdots \alpha_N} z_1^{\alpha_1} \cdots z_N^{\alpha_N}. \tag{2.1}$$

It will be convenient to use the following notation:

$$\mathbf{z} = (z_1, ..., z_N), \tag{2.2}$$

$$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N), \tag{2.3}$$

$$\mathbf{z}^{\boldsymbol{\alpha}} \equiv z_1^{\alpha_1} \cdots z_N^{\alpha_N}, \tag{2.4}$$

$$I_N = \{1, 2, ..., N\}, \tag{2.5}$$

and to denote the lattice set of vectors $\{\boldsymbol{\alpha}\}$, with nonnegative integer components, by S, i.e.,

$$S = \{\boldsymbol{\alpha} \mid \alpha_i \text{ a nonnegative integer}, i \in I_N\}. \tag{2.6}$$

The power series in Eq. (2.1) now may be written in the compact form

$$f(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in S} c_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}. \tag{2.7}$$

Rational approximants to $f(\mathbf{z})$ of the form

$$f_{m/m}(\mathbf{z}) = \frac{\sum_{\boldsymbol{\mu} \in S_m} a_{\boldsymbol{\mu}} \mathbf{z}^{\boldsymbol{\mu}}}{\sum_{\boldsymbol{\sigma} \in S_m} b_{\boldsymbol{\sigma}} \mathbf{z}^{\boldsymbol{\sigma}}}, \tag{2.8}$$

where

$$S_m = \{\boldsymbol{\alpha} \mid 0 \leqslant \alpha_i \leqslant m, i \in I_N\}, \tag{2.9}$$

have been defined and considered by Chisholm and McEwan [2]. We shall refer to these approximants as being diagonal rational approximants or Chisholm approximants (CA's). The coefficients $a_{\boldsymbol{\mu}}$ and $b_{\boldsymbol{\sigma}}$ are found by solving a set of defining equations given by equating coefficients in $f(\mathbf{z})$ and in the expansion of $f_{m/m}(\mathbf{z})$ up to a certain order.

In this paper, we shall consider general-off-diagonal rational approximants of the form

$$f_{\mathbf{m}/\mathbf{n}}(\mathbf{z}) = \frac{\sum_{\boldsymbol{\mu}\in S_{\mathbf{m}}} a_{\boldsymbol{\mu}}\mathbf{z}^{\boldsymbol{\mu}}}{\sum_{\boldsymbol{\sigma}\in S_{\mathbf{n}}} b_{\boldsymbol{\sigma}}\mathbf{z}^{\boldsymbol{\sigma}}}, \tag{2.10}$$

where $\mathbf{m}$ and $\mathbf{n}$ are points in the lattice set S, and where

$$S_{\mathbf{m}} = \{\boldsymbol{\alpha} \mid 0 \leqslant \alpha_i \leqslant m_i\,,\, i \in I_N\}, \tag{2.11}$$

$$S_{\mathbf{n}} = \{\boldsymbol{\alpha} \mid 0 \leqslant \alpha_i \leqslant n_i\,,\, i \in I_N\}. \tag{2.12}$$

Such an approximant may be referred to as the most general off-diagonal approximant, as the maximum power of each variable is different in the numerator and the denominator, and different variables may have different maximum powers. In certain circumstances one might restrict oneself to the simple-off-diagonal case in which $\mathbf{m} = (m,..., m)$ and $\mathbf{n} = (n,..., n)$, i.e., that in which $S_{\mathbf{m}}$ and $S_{\mathbf{n}}$ are hypercubes instead of hyper-rectangular-boxes. Such a restriction is convenient, for example, when $f(\mathbf{z})$ is a function symmetric in its variables.

There are

$$\prod_{i\in I_N} (m_i + 1) + \prod_{i\in I_N} (n_i + 1) \tag{2.13}$$

unknown coefficients in Eq. (2.10), and these coefficients must be given as the solution of a set of defining equations. These equations are, as for Padé approximants and Chisholm approximants, given by equating coefficients of $f(\mathbf{z})$ and $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ up to a certain order. It is obvious from Eq. (2.10) that the coefficients $a_{\boldsymbol{\mu}}$ and $b_{\boldsymbol{\sigma}}$ will be determined at most up to a common multiplicative factor. We shall thus assume that we can impose the normalization condition

$$b_{0,0,...,0} = b_{\mathbf{0}} = 1. \tag{2.14}$$

Such a normalization condition may not be possible in certain abnormal cases, for instance, if the defining equations only have a solution for which $b_{\mathbf{0}} = 0$. We shall not consider such cases in this paper.

By multiplying the difference between $f(\mathbf{z})$ and $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ by the denominator of $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ we obtain an equation of the form

$$\sum_{\boldsymbol{\sigma}\in S_{\mathbf{n}}} b_{\boldsymbol{\sigma}}\mathbf{z}^{\boldsymbol{\sigma}} \sum_{\boldsymbol{\alpha}\in S} c_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\mu}\in S_{\mathbf{m}}} a_{\boldsymbol{\mu}}\mathbf{z}^{\boldsymbol{\mu}} = \sum_{\boldsymbol{\beta}\in S} d_{\boldsymbol{\beta}}\mathbf{z}^{\boldsymbol{\beta}}. \tag{2.15}$$

Both sides of this equation may be "formal" power series.

The correct number of defining equations will be obtained by requiring

$$d_{\beta} = 0, \qquad \beta \in S_m \cup S_n, \tag{2.16}$$

$$d_{\beta} = 0, \qquad \beta \in S_3, \tag{2.17}$$

and

$$\sum_{\beta \in R_{4;p}} d_{\beta} = 0, \qquad p \in P, \tag{2.18}$$

where the sets S_3, $R_{4;p}$ and P (all subsets of S) will be defined below. S_3 will be a set bordering $S_m \cup S_n$. One requirement that we consider necessary is that $S_m \cup S_n \cup S_3$ is a set obeying the "rectangular rule." This rule may be stated in the form

$$\alpha \in S_m \cup S_n \cup S_3 \Rightarrow \beta \in S_m \cup S_n \cup S_3, \qquad \text{if} \quad \beta < \alpha, \tag{2.19}$$

where the partial ordering

$$\beta < \alpha, \qquad \text{means} \quad \beta \neq \alpha;\ \beta_i \leqslant \alpha_i,\ i \in I_N. \tag{2.20}$$

The rectangular rule is considered to be necessary, as it implies that when $f(\mathbf{z}) - f_{m/n}(\mathbf{z})$ has imposed on it the condition that the term $\mathbf{z}^{\alpha}$ is absent, then $\mathbf{z}^{\beta}$ is also absent for $\beta < \alpha$. The phrase "matching power series up to a certain order" when applied to functions of N-variables is thus taken to mean that the coefficients are equated in a region obeying the rectangular rule. The set $S_4 = \bigcup_{p \in P} R_{4;p}$ will be a set bordering $S_m \cup S_n \cup S_3$ and the set $S_m \cup S_n \cup S_3 \cup S_4$ will also be a set obeying the rectangular rule.

Equations (2.16)–(2.18) yield

$$\sum_{\sigma \in S_n} b_{\sigma} c_{\beta-\sigma} = a_{\beta}, \qquad \beta \in S_m, \tag{2.21}$$

$$\sum_{\sigma \in S_n} b_{\sigma} c_{\beta-\sigma} = 0, \qquad \beta \in (S_n \backslash S_m) \cup S_3, \tag{2.22}$$

and

$$\sum_{\beta \in R_{4;p}} \sum_{\sigma \in S_n} b_{\sigma} c_{\beta-\sigma} = 0, \qquad p \in P, \tag{2.23}$$

where we use the convention

$$c_{\alpha} \equiv 0, \qquad \text{if} \quad \alpha_i < 0 \text{ for at least one } i \text{ in } I_N. \tag{2.24}$$

The defining equations for the unknown coefficients a_{μ} and b_{σ} thus form a set of linear equations. Normally, the set of equations formed by (2.22), (2.23), and (2.14), ($b_0 = 1$), may be solved for the coefficients b_{σ}. The coefficients a_{μ}

then may be found directly from the equations in (2.21). The sets S_3, $R_{4;\mathbf{p}}$, and P will now be defined in a manner that will indicate that the number of linear equations equals the number of unknown coefficients. In defining these sets, a systematic method, the prong method, of solving the linear equations for the unknowns, will also be indicated and we shall later find necessary and sufficient conditions for the set of linear equations to be nondegenerate.

Let

$$S_1 = S_{\mathbf{m}} \cap S_{\mathbf{n}}\,; \qquad S_2 = (S_{\mathbf{m}} \cup S_{\mathbf{n}})\backslash S_1\,; \tag{2.25}$$

$$m_i' = \min(m_i, n_i); \tag{2.26}$$

$$n_i' = \max(m_i, n_i). \tag{2.27}$$

Then

$$S_1 = \{\boldsymbol{\alpha} \mid 0 \leqslant \alpha_i \leqslant m_i', i \in I_N\}. \tag{2.28}$$

The set P, a subset of S_1, is now a set of points with a certain amount of symmetry. Specifically,

$$P = \{\mathbf{p} \mid \mathbf{p} \in S_1\,;\, I_{\mathbf{p}} = \{j \mid p_j = \max_{i \in I_N} p_i\} \text{ has at least two elements}\}. \tag{2.29}$$

For each $\mathbf{p} \in P$, write $p = \max_{i \in I_N} p_i$. For each $\mathbf{p} \in P$, other subsets of S are defined as follows:

$$R_{1;\mathbf{p}} = \{\mathbf{p}\} \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} \mid p < \alpha_i \leqslant m_i';\, \alpha_j = p_j\,, j \neq i\} \tag{2.30}$$

$$R_{2;\mathbf{p}} = \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} \mid m_i' < \alpha_i \leqslant n_i';\, \alpha_j = p_j\,, j \neq i\} \tag{2.31}$$

$$R_{3;\mathbf{p}} = \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} \mid n_i' < \alpha_i \leqslant m_i' + n_i' - p;\, \alpha_j = p_j\,, j \neq i\} \tag{2.32}$$

$$R_{4,\mathbf{p}} = \bigcup_{i \in I_{\mathbf{p}}} \{\boldsymbol{\alpha} \mid \alpha_i = m_i' + n_i' - p + 1;\, \alpha_j = p_j\,, j \neq i\} \tag{2.33}$$

except that

$$R_{4;0} = \varnothing \qquad \text{(the empty set)}. \tag{2.34}$$

Write

$$S_3 = \bigcup_{\mathbf{p} \in P} R_{3;\mathbf{p}}\,, \tag{2.35}$$

$$S_4 = \bigcup_{\mathbf{p} \in P} R_{4;\mathbf{p}}\,, \tag{2.36}$$

and

$$R_{\mathbf{p}} = R_{1;\mathbf{p}} \cup R_{2;\mathbf{p}} \cup R_{3;\mathbf{p}} \cup R_{4;\mathbf{p}}\,. \tag{2.37}$$

The set of points $R_{\mathbf{p}}$ form a geometrical figure, called a prong, with l branches in the lattice space S, where l is the number of elements in $I_{\mathbf{p}}$. One of these branches is represented in Fig. 2.1.

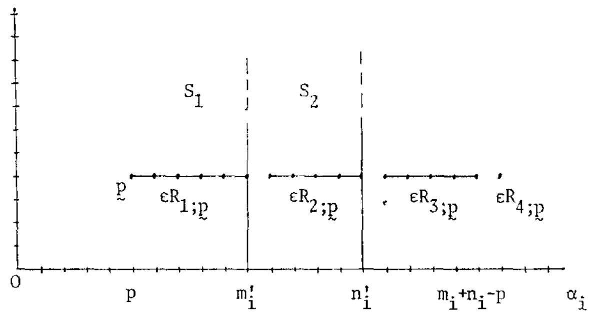

FIG. 2.1. One branch of a prong P .

It is easy to show that each point $\boldsymbol{\alpha} \in S_1$ lies in one and only one of the sets $R_{1;\mathbf{p}}$. Hence,

$$\bigcup_{\mathbf{p}\in P} R_{1;\mathbf{p}} = S_1 . \tag{2.38}$$

One can also check that

$$R_{\mathbf{p}_1} \cap R_{\mathbf{p}_2} = \varnothing, \qquad \text{for} \quad \mathbf{p}_1 \neq \mathbf{p}_2 . \tag{2.39}$$

It is possible to show that the number of linear equations is just sufficient to determine the unknown coefficients $a_{\boldsymbol{\mu}}$ and $b_{\boldsymbol{\sigma}}$. One may do this by looking at each of the sets $R_{\mathbf{p}}$. A coefficient $a_{\boldsymbol{\alpha}}$ or $b_{\boldsymbol{\alpha}}$ is said to be "attached" to $R_{\mathbf{p}}$ if $\boldsymbol{\alpha} \in R_{\mathbf{p}}$. Similarly, an equation $d_{\boldsymbol{\alpha}} = 0$ is said to be attached to $R_{\mathbf{p}}$ if $\boldsymbol{\alpha} \in R_{\mathbf{p}}$. Each point $\boldsymbol{\alpha} \in R_{1;\mathbf{p}}$ has two attached coefficients and each point $\boldsymbol{\alpha} \in R_{2;\mathbf{p}}$ has one attached coefficient. Hence, from (2.30) and (2.31), it may be seen that there are

$$2 + \sum_{i\in I_{\mathbf{p}}} (m_i + n_i - 2p)$$

coefficients attached to $R_{\mathbf{p}}$. The number of equations attached to $R_{\mathbf{p}}$ is also equal to this same number. An exception is $R_{\mathbf{0}}$, for which there is one less equation, but this is compensated by the normalization condition $b_{\mathbf{0}} = 1$.

In general, there may be points in S_2 that are not in the set $\bigcup_{\mathbf{p}\in P} R_{2;\mathbf{p}}$. Such points form the set

$$S_5 = S_2 \Big\backslash \bigcup_{\mathbf{p}\in P} R_{2;\mathbf{p}} . \tag{2.40}$$

For each point $\boldsymbol{\alpha} \in S_5$, there is one attached unknown coefficient, either $a_{\boldsymbol{\alpha}}$ or $b_{\boldsymbol{\alpha}}$, depending on whether $\boldsymbol{\alpha} \in S_{\mathbf{m}}$ or $\boldsymbol{\alpha} \in S_{\mathbf{n}}$, and one attached equation $d_{\boldsymbol{\alpha}} = 0$.

We have now proved the following theorem:

THEOREM 2.1. *The number of defining linear equations, given by* (2.21)–(2.23) *and* $b_{\mathbf{0}} = 1$, *equals the number of unknown coefficients,* (2.13), *in the rational approximant* (2.10).

The set of equations attached to $R_{\mathbf{p}}$ will be denoted by $E_{\mathbf{p}}$. We shall see in the later sections how the sets $E_{\mathbf{p}}$ may be given a suitable ordering such that on solving each set $E_{\mathbf{p}}$ in turn, the unknowns at any given stage are the coefficients attached to $R_{\mathbf{p}}$.

3. Approximants of Series in Two Variables

A study of two variable approximants will illustrate the definition of the general-off-diagonal rational approximants in N-variables, and also illustrate methods and theorems that will be generalized later to cover the N-variable case.

Let $f(z_1, z_2)$ be a function of two variables defined by its power series expansion. The definitions in the previous section then yield the following system:

$$f(z_1, z_2) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} c_{\alpha\beta} z_1^{\alpha} z_2^{\beta} = \sum_{(\alpha,\beta)\in S} c_{\alpha\beta} z_1^{\alpha} z_2^{\beta}, \tag{3.1}$$

where

$$S = \{(\alpha, \beta) \mid \alpha, \beta \text{ nonnegative integers}\}.$$

A general-off-diagonal rational approximant to f has the form

$$\begin{aligned} f_{\mathbf{m}/\mathbf{n}}(z_1, z_2) &= \frac{\sum_{(\mu,\nu)\in S_{\mathbf{m}}} a_{\mu\nu} z_1^{\mu} z_2^{\nu}}{\sum_{(\sigma,\tau)\in S_{\mathbf{n}}} b_{\sigma\tau} z_1^{\sigma} z_2^{\tau}} \\ &= f_{m_1,m_2/n_1,n_2}(z_1, z_2) = \frac{\sum_{\mu=0}^{m_1} \sum_{\nu=0}^{m_2} a_{\mu\nu} z_1^{\mu} z_2^{\nu}}{\sum_{\sigma=0}^{n_1} \sum_{\tau=0}^{n_2} b_{\sigma\tau} z_1^{\sigma} z_2^{\tau}}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} S_{\mathbf{m}} &= \{(\mu, \nu) \mid 0 \leqslant \mu \leqslant m_1, 0 \leqslant \nu \leqslant m_2\}, \\ S_{\mathbf{n}} &= \{(\sigma, \tau) \mid 0 \leqslant \sigma \leqslant n_1, 0 \leqslant \tau \leqslant n_2\}. \end{aligned}$$

There are $(m_1 + 1)(m_2 + 1) + (n_1 + 1)(n_2 + 1)$ coefficients in (3.2) to be found from a set of defining linear equations.

Equation (2.15) restricted to two variables becomes

$$\sum_{\sigma=0}^{n_1}\sum_{\tau=0}^{n_2} b_{\sigma\tau} z_1^{\sigma} z_2^{\tau} \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} c_{\alpha\beta} z_1^{\alpha} z_2^{\beta} - \sum_{\mu=0}^{m_1}\sum_{\nu=0}^{m_2} a_{\mu\nu} z_1^{\mu} z_2^{\nu} = \sum_{\gamma=0}^{\infty}\sum_{\delta=0}^{\infty} d_{\gamma\delta} z_1^{\gamma} z_2^{\delta}. \quad (3.3)$$

The definition of the set P given by (2.29),

$$P = \{\mathbf{p} \mid \mathbf{p} \in S_1 \,;\, I_{\mathbf{p}} = \{j \mid p_j = \max_{i \in I_N} p_i\} \text{ has at least two elements}\},$$

clearly yields the following set for two variables:

$$P = \{(p, p) \mid 0 \leqslant p \leqslant p_m\},$$

where

$$p_m = \min(m_1, m_2, n_1, n_2)$$

is the maximum allowable value for the parameter p. In the rest of this section, the parameter p will always be an integer between zero and p_m. For each $(p, p) = \mathbf{p} \in P$, Eqs. (2.30)–(2.34) yield the following sets:

$$R_{1;\mathbf{p}} = \{(p, p)\} \cup \{(\alpha, p) \mid p < \alpha \leqslant \min(m_1, n_1)\}$$
$$\cup \{(p, \beta) \mid p < \beta \leqslant \min(m_2, n_2)\};$$

$$R_{2;\mathbf{p}} = \{(\alpha, p) \mid \min(m_1, n_1) < \alpha \leqslant \max(m_1, n_1)\}$$
$$\cup \{(p, \beta) \mid \min(m_2, n_2) < \beta \leqslant \max(m_2, n_2)\};$$

$$R_{3;\mathbf{p}} = \{(\alpha, p) \mid \max(m_1, n_1) < \alpha \leqslant m_1 + n_1 - p\}$$
$$\cup \{(p, \beta) \mid \max(m_2, n_2) < \beta \leqslant m_2 + n_2 - p\};$$

$$R_{4;\mathbf{p}} = \{(m_1 + n_1 - p + 1, p), (p, m_2 + n_2 - p + 1)\},$$

except that $R_{4;0} = \varnothing$, the empty set.

The set of equations, $E_{\mathbf{p}} = E_{p,p}$, is made up of equations

$$d_{\alpha\beta} = 0, \qquad (\alpha, \beta) \in R_{1;\mathbf{p}} \cup R_{2;\mathbf{p}} \cup R_{3;\mathbf{p}}, \quad (3.4a)$$

and

$$d_{m_1+n_1-p+1,p} + d_{p,m_2+n_2-p+1} = 0, \qquad \text{if } p \neq 0. \quad (3.4b)$$

It is useful to consider a few diagrams in the two-dimensional lattice set S to illustrate the types of regions in which one requires $d_{\gamma\delta} = 0$, and the pairs of points on which one requires $\sum d_{\gamma\delta} = 0$. Figure 3.1 illustrates the situation for an example of a general-off-diagonal approximant. The various regions $S_{\mathbf{m}}$, $S_{\mathbf{n}}$, S_i ($i = 1, 2, 3, 4, 5$) and the points making up the prong $R_{\mathbf{p}}$ are indicated. Figure 3.2 illustrates the situation for a simple-off-diagonal approximant for which $m_1 = m_2 = m$, $n_1 = n_2 = n$. Figure 3.3 illustrates

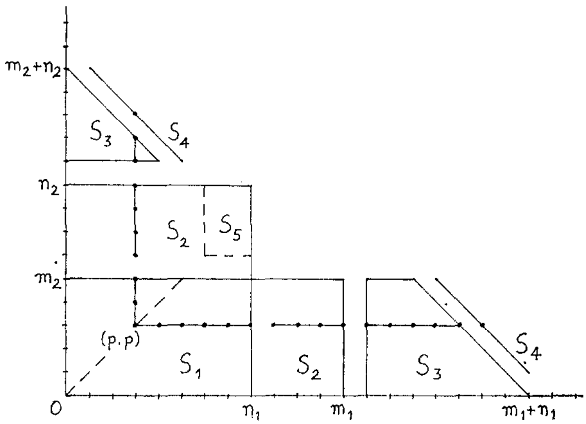

FIG. 3.1. GOD approximant for two variables.

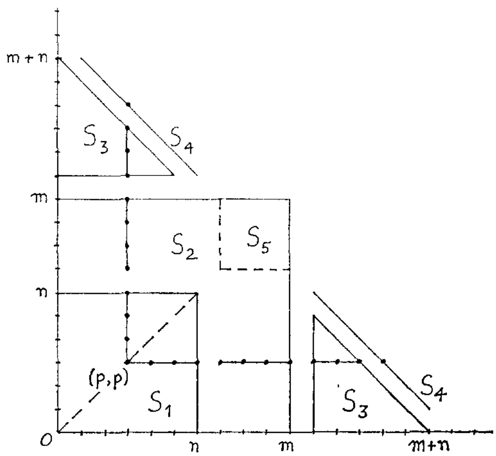

FIG. 3.2. SOD approximant for two variables.

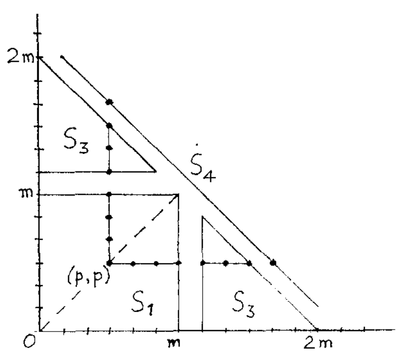

FIG. 3.3. CA for two variables.

the situation for a diagonal or Chisholm approximant. It would appear from the diagrams that the generalizations to the various off-diagonal cases is a fairly natural process.

The sets of equations $E_{\mathbf{p}}$, $\mathbf{p} \in P$, will now be solved in the order

$$E_{0,0}, E_{1,1}, E_{2,2}, \ldots, E_{p_m, p_m}.$$

This ordering of the equations yields a successive method of solving the defining equations for the unknown coefficients in the rational approximant. This method has been termed the prong method (Hughes Jones and Makinson, [5]). It enables the coefficients attached to each set $R_{\mathbf{p}}$ to be determined by the equations $E_{\mathbf{p}}$ attached to the same set.

We start with the normalization condition,

$$b_{0,0} = 1. \tag{3.5}$$

Now, consider the set $E_{0,0}$. Included in this set are the equations $d_{m_1+1,0} = 0, \ldots, d_{m_1+n_1,0} = 0$. These equations written in matrix form become, putting $b_{0,0} = 1$,

$$\begin{pmatrix} c_{m_1-n_1+1,0} & \cdots & c_{m_1,0} \\ \vdots & & \vdots \\ c_{m_1,0} & \cdots & c_{m_1+n_1-1,0} \end{pmatrix} \begin{pmatrix} b_{n_1,0} \\ \vdots \\ b_{1,0} \end{pmatrix} = \begin{pmatrix} -c_{m_1+1,0} \\ \vdots \\ -c_{m_1+n_1,0} \end{pmatrix}, \tag{3.6}$$

where we use the convention that $c_{\alpha\beta} = 0$ if either α or β is negative. The coefficients $b_{1,0}, \ldots, b_{n_1,0}$ will be uniquely determined if the matrix in this equation is nonsingular. Having found this solution, the coefficients $a_{0,0}, a_{1,0}, \ldots, a_{m_1,0}$ may be determined from the equations $d_{0,0} = 0, \ldots, d_{m_1,0} = 0$, i.e.,

$$a_{\alpha,0} = \sum_{\sigma=0}^{\min(n_1,\alpha)} b_{\sigma,0} c_{\alpha-\sigma,0}, \qquad \alpha = 0, 1, \ldots, m_1. \tag{3.7}$$

Equations (3.5)–(3.7) are precisely the equations that define the $[m_1/n_1]$ Padé approximant to the series

$$g(z_1) \equiv f(z_1, 0) = \sum_{\alpha=0}^{\infty} c_{\alpha 0} z_1^{\alpha}.$$

The coefficients $b_{0,0}, b_{0,1}, \ldots, b_{0,n_2}$ and $a_{0,0}, a_{0,1}, \ldots, a_{0,m_2}$ are similarly defined by the equations that determine the $[m_2/n_2]$ Padé approximant to

$$h(z_2) \equiv f(0, z_2) = \sum_{\beta=0}^{\infty} c_{0\beta} z_2^{\beta}.$$

By referring to (3.2), it may be seen that we have proved the following theorem:

THEOREM 3.1. *When* $z_2 = 0$, *the approximant* $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ *reduces to* $g_{m_1/n_1}(z_1)$, *the* $[m_1/n_1]$ *Pade approximant to* $g(z_1) \equiv f(z_1, 0)$; *and when* $z_1 = 0$, $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ *reduces to* $h_{m_2/n_2}(z_2)$, *the* $[m_2/n_2]$ *Pade approximant to* $h(z_2) \equiv f(0, z_2)$.

Let us now assume that the sets of equations

$$E_{0,0}, E_{1,1}, \ldots, E_{p-1,p-1}$$

have been considered and the corresponding attached coefficients

$$\begin{aligned}
&a_{\mu\nu}, && \mu < p,\ 0 \leqslant \nu \leqslant m_2, \\
&a_{\mu\nu}, && \nu < p,\ 0 \leqslant \mu \leqslant m_1, \\
&b_{\sigma\tau}, && \sigma < p,\ 0 \leqslant \tau \leqslant n_2, \\
&b_{\sigma\tau}, && \tau < p,\ 0 \leqslant \sigma \leqslant n_1,
\end{aligned}$$

have been determined. Consider the set of equations $E_{p,p}$. It may be seen that the equations in this set that do not involve the coefficients $a_{\mu\nu}$ may be written out in the matrix form

$$\begin{pmatrix}
c_{m_1-n_1+1,0} & \cdots & c_{m_1-p,0} & & & & c_{m_1-p+1,0} \\
\vdots & & \vdots & & \bigcirc & & \vdots \\
c_{m_1-p,0} & \cdots & c_{m_1+n_1-2p-1,0} & & & & c_{m_1+n_1-2p,0} \\
 & & & c_{0,m_2-n_2+1} & \cdots & c_{0,m_2-p} & c_{0,m_2-p+1} \\
 & \bigcirc & & \vdots & & \vdots & \vdots \\
 & & & c_{0,m_2-p} & \cdots & c_{0,m_2+n_2-2p-1} & c_{0,m_2+n_2-2p} \\
c_{m_1-p+1,0} & \cdots & c_{m_1+n_1-2p,0} & c_{0,m_2-p+1} & \cdots & c_{0,m_2+n_2-2p} & c_{m_1+n_1-2p+1,0} + c_{0,m_2+n_2-2p+1}
\end{pmatrix}
\begin{pmatrix}
b_{n_1,p} \\ \vdots \\ b_{p+1,p} \\ b_{p,n_2} \\ \vdots \\ b_{p,p+1} \\ b_{p,p}
\end{pmatrix}$$

$$= \text{quantity involving known coefficients.} \tag{3.8}$$

It will be convenient for future use to denote the matrix in this equation by $D_\mathbf{p}$, and the vector of the $b_{\sigma\tau}$ by $\mathbf{b}_\mathbf{p}$. The coefficients $b_{\sigma\tau}$ attached to $R_{p,p}$ may be uniquely determined from this equation if the matrix is nonsingular. The coefficients $a_{\mu\nu}$ attached to $R_{p,p}$ then may be found directly from the rest of the equations in $E_{p,p}$, i.e., from

$$d_{p,p} = 0, \quad d_{p+1,p} = 0, \ldots, \quad d_{m_1,p} = 0, \quad d_{p,p+1} = 0, \ldots, \quad d_{p,m_2} = 0.$$

By induction, all coefficients $a_{\mu\nu}$ and $b_{\sigma\tau}$ attached to the sets $R_{p,p}$,

$0 \leqslant p \leqslant p_m$, may be found by the above procedure. The remaining coefficients left to determine are those attached to the region

$$S_5 = S_2 \Big\backslash \Big(\bigcup_{\mathbf{p} \in P} R_{2;\mathbf{p}} \Big).$$

The coefficients attached to S_5 may be determined one at a time by suitably ordering the equations attached to S_5. Any ordering of the equations such that $d_{\gamma\delta} = 0$ is considered only if all $d_{\alpha\beta} = 0$, $\alpha \leqslant \gamma$, $\beta \leqslant \delta$, $(\alpha, \beta) \neq (\gamma, \delta)$, have already been considered is sufficient. A suitable ordering for the example shown in Fig. 3.1 is

$$\begin{array}{lll} d_{p_m+1,p_m+1} = 0, \ldots, & d_{n_1,p_m+1} = 0, \\ \vdots & \vdots \\ d_{p_m+1,n_2} = 0, \ldots, & d_{n_1,n_2} = 0. \end{array}$$

These equations will determine the unknown coefficients if $c_{00} \neq 0$. The region S_5 has unknown coefficients $b_{\sigma\tau}$ attached to it if

$$\min(n_1, n_2) > p_m = \min(m_1, m_2, n_1, n_2), \tag{3.9}$$

and then the condition $c_{00} \neq 0$ is necessary for unique determination of the coefficients. If the condition in (3.9) does not hold then S_5 (if nonempty) will have unknown coefficients $a_{\mu\nu}$ attached to it and these will be determined immediately once all the equations in the sets $E_{\mathbf{p}}$, $\mathbf{p} \in P$, have been solved.

It is useful to introduce the following notation:

$$C_{p;1} = \begin{pmatrix} c_{m_1-n_1+1,0} & \cdots & c_{m_1-p,0} \\ \vdots & & \vdots \\ c_{m_1-p,0} & \cdots & c_{m_1+n_1-2p-1,0} \end{pmatrix} \tag{3.10}$$

$$X_{p;1} = (c_{m_1-p+1,0} \quad \cdots \quad c_{m_1+n_1-2p,0})^T, \tag{3.11}$$

$$Y_{p;1} = c_{m_1+n_1-2p+1,0}\,. \tag{3.12}$$

Similarly, $C_{p;2}$, $X_{p;2}$, and $Y_{p;2}$ will denote the corresponding quantities when the second index in the $c_{\alpha\beta}$ is nonzero and the first index is zero, e.g.,

$$Y_{p;2} = c_{0,m_2+n_2-2p+1}\,.$$

The matrix $D_{\mathbf{p}}$ of Eq. (3.8) is then of the form

$$D_{\mathbf{p}} = \begin{pmatrix} C_{p;1} & 0 & X_{p;1} \\ 0 & C_{p;2} & X_{p;2} \\ X_{p;1}^T & X_{p;2}^T & Y_{p;1} + Y_{p;2} \end{pmatrix}. \tag{3.13}$$

Such matrices occur for $p = 1, 2, \ldots, p_m$. For $p = 0$, let

$$D = \begin{pmatrix} C_{0;1} & 0 & X_{0;1} \\ 0 & C_{0;2} & X_{0;2} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

The complete set of equations for all the $b_{\sigma\tau}$ then may be written in the lower triangular block form

$$\begin{bmatrix} D_0 & & & & & & & \\ \vdots & D_1 & & & & \bigcirc & & \\ \vdots & & \ddots & & & & & \\ \vdots & & & D_p & & & & \\ \vdots & & & & \ddots & & & \\ \vdots & & & & & D_{p_m} & & \\ \vdots & & & & & & c_{00} & \\ \vdots & & & & & & & \ddots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & c_{00} \end{bmatrix} \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_p \\ \vdots \\ \mathbf{b}_{p_m} \\ b_{p_m+1, p_m+1} \\ \vdots \\ b_{n_1, n_2} \end{bmatrix} = \mathbf{u}_{m/n}, \tag{3.15}$$

where $\mathbf{u}_{m/n}$ is a column vector of dimension $(n_1 + 1) \times (n_2 + 1)$ with unity in the $(n_1 + n_2 + 1)$th place and zeros elsewhere. The matrix will terminate with D_{p_m} unless $\min(n_1, n_2) > p_m$.

The prong method of solving the set of defining equations for the rational approximant in Eq. (3.2) thus yields the following theorem:

THEOREM 3.2. *The approximant $f_{m/n}(z_1, z_2)$ is determined by a nondegenerate set of linear equations if and only if*

(i) *$C_{0;1}$ and $C_{0;2}$ are nonsingular matrices,*

(ii) *D_p, $p = 1, 2, \ldots, p_m$ are nonsingular matrices,*

(iii) *$c_{00} \neq 0$, if $\min(n_1, n_2) > p_m = \min(m_1, m_2, n_1, n_2)$.*

An approximant that is given by a nondegenerate set of equations may be referred to as a nondegenerate approximant. It is of interest to note that the conditions for nondegeneracy of $f_{m/n}(z_1, z_2)$ involve only the coefficients $c_{\alpha 0}$ and $c_{0\beta}$.

It is important to remember that the $c_{\alpha 0}(c_{0\beta})$ that occur in the $C_{p;1}(C_{p;2})$ always have increasing $\alpha(\beta)$ on moving to the right or downwards in the matrix $C_{p;1}(C_{p;2})$. If this cannot occur then the matrix becomes void. For instance if $n_1 = p_m$ and $n_2 > p_m$, then

$$D_{\mathbf{p}_m} = \begin{pmatrix} C_{p_m;2} & X_{p_m;2} \\ X^T_{p_m;2} & Y_{p_m;1} + Y_{p_m;2} \end{pmatrix},$$

i.e., $C_{p_m;1}$ and $X_{p_m;1}$ have become void since

$$m_1 - n_1 + 1 > m_1 - p_m > m_1 + n_1 - 2p_m - 1$$

and

$$m_1 - p_m + 1 > m_1 + n_1 - 2p_m .$$

Such cases will not occur for $p < p_m$.

The quantities defined in Eqs. (3.10)–(3.13) satisfy certain recurrence relationships. For instance

$$C_{p-1;i} = \begin{pmatrix} C_{p;i} & X_{p;i} \\ X^T_{p;i} & Y_{p;i} \end{pmatrix}, \qquad i = 1, 2. \tag{3.16}$$

If one is interested in generating approximants successively, then the following types of properties would be useful:

$$C(m_i + 1, n_i + 1; p + 1; i) = C(m_i, n_i; p; i), \qquad i = 1, 2,$$
$$D(m_1 + 1, m_2 + 1, n_1 + 1, n_2 + 1; \mathbf{p} + 1) = D(m_1, m_2, n_1, n_2; \mathbf{p}),$$

where the notation has been enlarged so that the dependence on the order of the approximant is included. The methods for generating Chisholm approximants developed in a previous paper, Hughes Jones and Makinson [5], may easily be extended to off-diagonal approximants.

The conditions for nondegeneracy of $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ involve the coefficients in the power series

$$g(z_1) \equiv f(z_1, 0) \qquad \text{and} \qquad h(z_2) \equiv f(0, z_2).$$

The matrices that are required to be nonsingular may be seen to be related to the matrices that have to be nonsingular when $g(z_1)$ and $h(z_2)$ are "normal" series, see for example Wall [6]. It is, therefore, of interest to find the connection between the normality of the series $g(z_1)$ and $h(z_2)$ and the normality of the series $f(z_1, z_2)$. The series $f(z_1, z_2)$ will be referred to as a normal series if the approximants $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ are all nondegenerate. It will be found that the necessary conditions are almost, but not quite, the same.

Let us assume for the moment that $g(z_1)$ and $h(z_2)$ are normal series. It is obvious from Theorem 3.1 that this condition at least is necessary for the normality of the series $f(z_1, z_2)$. The matrices $C_{p;i}$, $0 \leqslant p \leqslant p_m$, $i = 1, 2$, are now all nonsingular. It will be useful to find unit lower triangular matrices $L_{p;i}$ such that $L_{p;i}C_{p;i}$ are upper triangular matrices. To do this, all the leading minors of the $C_{p;i}$ must be nonsingular. This will not be true if $m_i - n_i + 1 < 0$ since the matrices $C_{p;i}$ then will have a number of zeros in the top left-hand corner. Consider the case $m_1 - n_1 + 1 < 0$. Then,

$$C_{p;1} = \begin{pmatrix} \bigcirc & . & c_{00} & c_{10} & \cdots & c_{m_1-p,0} \\ & \cdot & \vdots & \vdots & & \vdots \\ c_{00} & \cdots & c_{n_1-m_1-1,0} & c_{n_1-m_1,0} & \cdots & c_{n_1-p-1,0} \\ c_{10} & \cdots & c_{n_1-m_1,0} & c_{n_1-m_1+1,0} & \cdots & c_{n_1-p,0} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{m_1-p,0} & \cdots & c_{n_1-p-1,0} & c_{n_1-p,0} & \cdots & c_{m_1+n_1-2p-1,0} \end{pmatrix}$$

It is, however, an easy task to produce other matrices $\tilde{C}_{p;1}$, from the $C_{p;1}$, which have all their leading minors nonsingular. This may be done by reversing the order of the first $n_1 - m_1$ rows. Therefore, we let

$$\tilde{C}_{p;1} = \begin{pmatrix} c_{00} & .\cdots & c_{n_1-m_1-1,0} & c_{n_1-m_1,0} & \cdots & c_{n_1-p-1,0} \\ & \cdot & \vdots & \vdots & & \vdots \\ \bigcirc & & c_{00} & c_{10} & \cdots & c_{m_1-p,0} \\ c_{10} & \cdots & c_{n_1-m_1,0} & c_{n_1-m_1+1,0} & \cdots & c_{n_1-p,0} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{m_1-p,0} & \cdots & c_{n_1-p-1,0} & c_{n_1-p,0} & \cdots & c_{m_1+n_1-2p-1,0} \end{pmatrix}, \tag{3.17a}$$

if $m_1 - n_1 + 1 < 0$, and

$$\tilde{C}_{p;1} = C_{p;1}, \qquad \text{if} \quad m_1 - n_1 + 1 \geqslant 0. \tag{3.17b}$$

These matrices will all be nonsingular if $g(z_1)$ is a normal series, and all the leading minors will also be nonsingular. Let matrices $\tilde{C}_{p;2}$ be defined in a similar manner. Also define

$$\tilde{X}_{p;1} = (c_{n_1-p,0} \cdots c_{m_1-p+1,0} c_{n_1-p+1,0} \cdots c_{m_1+n_1-2p,0})^T,$$
$$\text{if} \quad m_1 - n_1 + 1 < 0, \tag{3.18a}$$

and

$$\tilde{X}_{p;1} = X_{p;1}, \qquad \text{if} \quad m_1 - n_1 + 1 \geqslant 0, \tag{3.18b}$$

and let quantities $\tilde{X}_{p;2}$ be defined in a similar manner.

The matrices that have to be reduced to upper triangular form in solving the set of defining equations for $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ are $\tilde{C}_{0;1}$, $\tilde{C}_{0;2}$, and

$$\tilde{D}_{\mathbf{p}} = \begin{pmatrix} \tilde{C}_{p;1} & 0 & \tilde{X}_{p;1} \\ 0 & \tilde{C}_{p;2} & \tilde{X}_{p;2} \\ X^T_{p;1} & X^T_{p;2} & Y_{p;1} + Y_{p;2} \end{pmatrix}, \qquad p = 1, 2, \ldots, p_m. \tag{3.19}$$

Let $L_{p;i}$ be the (unique) unit lower triangular matrix that reduces $\tilde{C}_{p;i}$ to upper triangular form. That is, $L_{p;i}\tilde{C}_{p;i}$ is an upper triangular matrix. The recurrence relation

$$\tilde{C}_{p-1;i} = \begin{pmatrix} \tilde{C}_{p;i} & \tilde{X}_{p;i} \\ X^T_{p;i} & Y_{p;i} \end{pmatrix}$$

immediately implies a recurrence relation of the form

$$L_{p-1;i} = \begin{pmatrix} L_{p;i} & 0 \\ W_{p;i} & 1 \end{pmatrix}. \tag{3.20}$$

Also, the unit lower triangular matrices $L_\mathbf{p}$ that reduce the $\tilde{D}_\mathbf{p}$ to upper triangular form are given by

$$L_0 = \begin{pmatrix} L_{0;1} & 0 & 0 \\ 0 & L_{0;2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$L_\mathbf{p} = \begin{pmatrix} L_{p;1} & 0 & 0 \\ 0 & L_{p;2} & 0 \\ W_{p;1} & W_{p;2} & 1 \end{pmatrix}, \qquad p = 1, 2,..., p_m . \tag{3.21}$$

It may thus be seen that all the information required to reduce to upper triangular form all the matrices that occur in the defining equations for $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ is contained in $L_{0;1}$ and $L_{0;2}$. Thus, the amount of actual elimination necessary when finding $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ is the same as that required when finding $g_{m_1/n_1}(z_1)$ and $h_{m_2/n_2}(z_2)$.

We are now in a position to prove a theorem giving sufficient conditions for the nondegeneracy of $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$.

THEOREM 3.3. *The approximant $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ is nondegenerate if*

(i) *$g(z_1)$ and $h(z_2)$ are normal series,*

(ii) $\Omega_p = W_{p;1}\tilde{X}_{p;1} + W_{p;2}\tilde{X}_{p;2} + Y_{p;1} + Y_{p;2} \neq 0,$ (3.22)

$$p = 1, 2,..., p_m .$$

Proof. The first condition ensures that the matrices $\tilde{C}_{p;i}$, $p = 0, 1,..., p_m$; $i = 1, 2$, are nonsingular.

The second condition ensures that the $\tilde{D}_\mathbf{p}$ are nonsingular, since by matrix multiplication we obtain

$$L_\mathbf{p}\tilde{D}_\mathbf{p} = \begin{pmatrix} L_{p;1}\tilde{C}_{p;1} & 0 & L_{p;1}\tilde{X}_{p;1} \\ 0 & L_{p,2}\tilde{C}_{p;2} & L_{p;2}\tilde{X}_{p;2} \\ 0 & 0 & \Omega_p \end{pmatrix},$$

where Ω_p is given by (3.22), and this matrix is only nonsingular if $\Omega_p \neq 0$.

Condition (ii) in Theorem 3.3 involves the coefficients $c_{\alpha 0}$ and $c_{0\beta}$. If during the calculation of a particular approximant one found that some Ω_p vanished, then we could overcome this difficulty by scaling in one of the variables. It is well known that Padé approximants are invariant under scaling, i.e., under $z \to Kz$. The approximants considered here, however, are not usually invariant under scaling in one variable only. This occurs because of the equations $\sum d_{\gamma\delta} = 0$, where the summation is over points in $R_{4;p}$. It is useful to consider scaling in one variable only as it enables us to prove an extension of a theorem proved in a previous paper, Hughes Jones and Makinson [5].

THEOREM 3.4. *The approximant* $\hat{f}_{m/n}(z_1, z_2)$ *to the series* $\hat{f}(z_1, z_2) = f(z_1, Kz_2)$ *will be nondegenerate if*

(i) $g(z_1)$ *and* $h(z_2)$ *are normal series,*

(ii) *the scale factor K is not equal to one of a finite number of values.*

Proof. The series $\hat{f}(z_1, z_2)$ is defined by

$$\hat{f}(z_1, z_2) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} K^{\beta} c_{\alpha\beta} z_1^{\alpha} z_2^{\beta}. \tag{3.23}$$

It is always assumed, of course, that $K \neq 0$.

Applying Theorem 3.2 to this series we find, after some row and column multiplications by powers of K, and some row interchanges if necessary to form the matrices $\tilde{C}_{p;i}$, that $\hat{f}_{m/n}(z_1, z_2)$ is nondegenerate if and only if

$$\tilde{C}_{0;1}, \qquad \tilde{C}_{0;2},$$

and

$$\tilde{\hat{D}}_p = \begin{pmatrix} \tilde{C}_{p;1} & 0 & \tilde{X}_{p;1} \\ 0 & K_p \tilde{C}_{p;2} & K_p \tilde{X}_{p;2} \\ X_{p;1}^T & K_p X_{p;2}^T & Y_{p;1} + K_p Y_{p;2} \end{pmatrix}, \qquad p = 1, 2,..., p_m, \tag{3.24}$$

are all nonsingular matrices, where

$$K_p = K^{m_2+n_2-2p+1}. \tag{3.25}$$

It is easy to see that the unit lower triangular matrices that reduce the above matrices to upper triangular form are identical to those found for the approximant $f_{m/n}(z_1, z_2)$. By matrix multiplication we find

$$L_p \tilde{\hat{D}}_p = \begin{pmatrix} L_{p;1} \tilde{C}_{p;1} & 0 & L_{p;1} \tilde{X}_{p;1} \\ 0 & K_p L_{p;2} \tilde{C}_{p;2} & K_p L_{p;2} \tilde{X}_{p;2} \\ 0 & 0 & \hat{\Omega}_p \end{pmatrix},$$

where

$$\hat{\Omega}_p = W_{p;1}\tilde{X}_{p;1} + K_p W_{p;2}\tilde{X}_{p;2} + Y_{p;1} + K_p Y_{p;2} . \tag{3.26}$$

The scalars $\hat{\Omega}_p$ will all be nonzero if

$$K_p \neq -\frac{W_{p;1}\tilde{X}_{p;1} + Y_{p;1}}{W_{p;2}\tilde{X}_{p;2} + Y_{p;2}}, \qquad p = 1, 2,..., p_m . \tag{3.27}$$

The numerator (denominator) of the right-hand side depends only on the $c_{\alpha 0}(c_{0\beta})$ and is nonzero if $g(z_1)(h(z_2))$ is a normal series.

By allowing **m** and **n** to take all possible values, we may prove another theorem:

THEOREM 3.5. *The series* $\hat{f}(z_1, z_2)$ *is a normal series if and only if*

(i) $g(z_1)$ *and* $h(z_1)$ *are normal series,*

(ii) *the scale factor K is not equal to one of a denumerable number of values.*

In a previous paper (Graves–Morris, Hughes Jones, and Makinson [4]) simple-off-diagonal approximants to a series in two variables have been defined. These are useful approximants to consider when $f(z_1, z_2)$ is a symmetric function in its variables. There are a number of theorems that may be proved fairly easily for such functions. If $m_1 = m_2 = m$ and $n_2 = n_1 = n$, then we denote the (simple-off-diagonal) approximant by $f_{m/n}(z_1, z_2)$.

If $f(z_1, z_2)$ is a symmetric function then, in particular, $c_{\alpha 0} = c_{0\alpha}$. Hence, when $m_1 = m_2$, $n_1 = n_2$, we have

$$C_{p;1} = C_{p;2} .$$

THEOREM 3.6. *If* $f(z_1, z_2)$ *is a symmetric function, then* $f_{m/n}(z_1, z_2)$ *is nondegenerate if and only if*

(i) $C_{p;1}$, $p = 0, 1, 2,..., p_m - 1$, *are all nonsingular matrices,*

(ii) $c_{00} \neq 0$, *if* $p_m < n$, *where* $p_m = \min(m, n)$.

Proof. The proof follows immediately from Theorem 3.2 and the fact that $C_{p;1} = C_{p;2}$.

THEOREM 3.7. *If* $f(z_1, z_2)$ *is a symmetric function and* $f_{m/n}(z_1, z_2)$ *is nondegenerate, then* $f_{m/n}(z_1, z_2)$ *is a symmetric function.*

Proof. This theorem follows by considering, during the prong method of solving the defining equations, the difference between the equations

$d_{m+1,p} = 0,\dots, d_{m+n-p,p} = 0$, and the equations $d_{p,m+1} = 0,\dots, d_{p,m+n-p} = 0$. We will then obtain, assuming that coefficients $b_{\sigma\tau}$ already calculated are symmetric, the equations

$$\begin{pmatrix} c_{m-n+1,0} & \cdots & c_{m-p,0} \\ \vdots & & \vdots \\ c_{m-p,0} & \cdots & c_{m+n-2p-1,0} \end{pmatrix} \begin{pmatrix} b_{n,p} & - & b_{p,n} \\ \vdots & & \vdots \\ b_{p+1,p} & - & b_{p,p+1} \end{pmatrix} = 0.$$

As the matrix is nonsingular, we deduce that the coefficients $b_{\sigma\tau}$ attached to R_p are symmetric. Hence, by induction, $b_{\alpha\beta} = b_{\beta\alpha}$ for all α, β. Also, $a_{\alpha\beta} = a_{\beta\alpha}$ since the defining equations are set up in a symmetric manner.

The following four theorems will be stated without proofs as proofs may be easily constructed.

THEOREM 3.8. *If $f(z_1, z_2)$ is a symmetric function, then $f_{m/n}(z_1, z_2)$ is nondegenerate if and only if*

(i) $g_{m-p/n-p}(z_1)$, $p = 0, 1,\dots, p_m - 1$, *are all nondegenerate Padé approximants,*

(ii) $c_{00} \neq 0$, *if* $p_m < n$.

THEOREM 3.9. *If $f(z_1, z_2)$ is a symmetric function, then all simple-off-diagonal approximants are nondegenerate if and only if $g(z_1)$ is a normal series.*

THEOREM 3.10. *If $f(z_1, z_2)$ is a symmetric function; $\hat{f}(z_1, z_2) = f(z_1, Kz_2)$; and $f_{m/n}(z_1, z_2)$ is nondegenerate, then $\hat{f}_{m/n}(z_1, z_2)$ is nondegenerate so long as*

$$K_p = K^{m+n-2p+1} \neq -1, \qquad p = 1, 2,\dots, p_m.$$

THEOREM 3.11. *If the conditions of Theorem 3.10 hold then we have invariance under scaling in one variable, i.e.,*

$$\hat{f}_{m/n}(z_1, z_2) = f_{m/n}(z_1, Kz_2).$$

A theorem of Common and Graves–Morris [3] concerning Chisholm approximants to a function of two variables that factorizes into the product of two functions of one variable may be easily extended to the general-off-diagonal approximants.

THEOREM 3.12. *If $f(z_1, z_2)$ is a function such that*

$$f(z_1, z_2) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} g_\alpha h_\beta z_1{}^\alpha z_2{}^\beta, \tag{3.28}$$

and $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ *is nondegenerate, then*

$$f_{\mathbf{m}/\mathbf{n}}(z_1, z_2) = g_{m_1/n_1}(z_1)\, h_{m_2/n_2}(z_2), \tag{3.29}$$

where $g_{m_1/n_1}(z_1)$ *is the* $[m_1/n_1]$ *Pade approximant to*

$$g(z_1) = \sum_{\alpha=0}^{\infty} g_\alpha z_1{}^\alpha,$$

and $h_{m_2/n_2}(z_2)$ *is the* $[m_2/n_2]$ *Pade approximant to*

$$h(z_2) = \sum_{\beta=0}^{\infty} h_\beta z_2{}^\beta.$$

Proof. If $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ is nondegenerate, then

$$g_{m_1/n_1}(z_1) = \frac{g_{m_1}^N(z_1)}{g_{n_1}^D(z_1)} \quad \text{and} \quad h_{m_2/n_2}(z_2) = \frac{h_{m_2}^N(z_2)}{h_{n_2}^D(z_2)}$$

are nondegenerate Padé approximants, where we have labeled the numerator and denominator polynomials in the Padé approximants with indices N and D. From the theory of Padé approximants

$$g_{n_1}^D(z_1)\, g(z_1) = g_{m_1}^N(z_1) + O(z_1^{m_1+n_1+1}),$$

$$h_{n_2}^D(z_2)\, h(z_2) = h_{m_2}^N(z_2) + O(z_2^{m_2+n_2+1}).$$

Hence, on multiplying both sides,

$$g_{n_1}^D(z_1)\, h_{n_2}^D(z_2) f(z_1, z_2) = g_{m_1}^N(z_1)\, h_{m_2}^N(z_2) + O(z_1^{m_1+n_1+1}) + O(z_2^{m_2+n_2+1}).$$

On inspecting Eq. (3.3) and Fig. 3.1, we see that this equation more than satisfies the defining equations for $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$. Since the coefficients in $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ are uniquely determined when the approximant is nondegenerate, we may deduce that

$$f_{m/n}(z_1, z_2) = g_{m_1/n_1}(z_1)\, h_{m_2/n_2}(z_2).$$

The final theorem that will be proved in this section involves the inverse function. This will be an extension of the theorem for Chisholm approximants proved by Chisholm [1].

THEOREM 3.13. *Let* $f(z_1, z_2)$ *be a series with an inverse series* $g(z_1, z_2)$ *in the sense that*

$$f(z_1, z_2)\, g(z_1, z_2) = 1. \tag{3.30}$$

If $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ *and* $g_{\mathbf{n}/\mathbf{m}}(z_1, z_2)$ *are nondegenerate then*

$$f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)\, g_{\mathbf{n}/\mathbf{m}}(z_1, z_2) = 1.$$

Proof. The approximant $f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)$ is defined by Eqs. (3.2)–(3.4). Notice also that the sets $R_{1;\mathbf{p}}$, $R_{2;\mathbf{p}}$, $R_{3;\mathbf{p}}$, and $R_{4;\mathbf{p}}$ are invariant under the interchange of $\mathbf{m}$ and $\mathbf{n}$.

If (3.2) is multiplied by

$$g(z_1, z_2) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} e_{\alpha\beta} z_1^{\alpha} z_2^{\beta},$$

then we obtain

$$\sum_{\sigma=0}^{n_1} \sum_{\tau=0}^{n_2} b_{\sigma\tau} z_1^{\sigma} z_2^{\tau} - \sum_{\mu=0}^{m_1} \sum_{\nu=0}^{m_2} a_{\mu\nu} z_1^{\mu} z_2^{\nu} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} e_{\alpha\beta} z_1^{\alpha} z_2^{\beta} = \sum_{\gamma=0}^{\infty} \sum_{\delta=0}^{\infty} \tilde{d}_{\gamma\delta} z_1^{\gamma} z_2^{\delta}, \tag{3.31}$$

where

$$\tilde{d}_{\gamma\delta} = \sum_{\alpha=0}^{\gamma} \sum_{\beta=0}^{\delta} d_{\gamma-\alpha,\delta-\beta} e_{\alpha\beta}.$$

Since the set $S_1 \cup S_2 \cup S_3$ obeys the rectangular rule it may be deduced that

$$d_{\gamma\delta} = 0, \qquad (\gamma, \delta) \in S_1 \cup S_2 \cup S_3 \Rightarrow \tilde{d}_{\gamma\delta} = 0,\ (\gamma, \delta) \in S_1 \cup S_2 \cup S_3.$$

This result, combined with the fact that the set $S_1 \cup S_2 \cup S_3 \cup S_4$ obeys the rectangular rule, may be seen to imply that

$$d_{\gamma_1,\delta_1} + d_{\gamma_2,\delta_2} = 0 \Rightarrow \tilde{d}_{\gamma_1,\delta_1} + \tilde{d}_{\gamma_2,\gamma_2}, \qquad \{(\gamma_1, \delta_1), (\gamma_2, \delta_2)\} = R_{4;\mathbf{p}}.$$

It now may be seen that (3.31), together with the above requirements for the vanishing of the coefficients $\tilde{d}_{\gamma\delta}$, is the defining equation for

$$g_{\mathbf{n}/\mathbf{m}}(z_1, z_2) = \frac{\sum_{\sigma=0}^{n_1} \sum_{\tau=0}^{n_2} b_{\sigma\tau} z_1^{\sigma} z_2^{\tau}}{\sum_{\mu=0}^{m_1} \sum_{\nu=0}^{m_2} a_{\mu\nu} z_1^{\mu} z_2^{\nu}}.$$

Hence,

$$f_{\mathbf{m}/\mathbf{n}}(z_1, z_2)\, g_{\mathbf{n}/\mathbf{m}}(z_1, z_2) = 1.$$

4. Approximants of Series in N Variables

In the previous section, methods and theorems for rational approximants to series in two variables were examined in some detail. The prong method for solving the defining equations gave a number of useful theorems about the conditions for nondegeneracy. The essence of the prong method lay in ordering the prongs so that, on solving the sets of equations $E_{\mathfrak{p}}$ in that order, i.e., $E_{0,0}$, $E_{1,1}, \ldots, E_{p_m,p_m}$, the coefficients $b_{\sigma\tau}$ and $a_{\mu\nu}$ could be found in a systematic manner. In this section, it will be shown how an ordering of the prongs $R_{\mathfrak{p}}$ for the case of N-variables may be introduced so that the coefficients $b_{\boldsymbol{\sigma}}$ and $a_{\boldsymbol{\mu}}$ may be found in a systematic manner. Of course, solving for the $a_{\boldsymbol{\mu}}$ is very straightforward, as they are determined immediately once the necessary $b_{\boldsymbol{\sigma}}$ have been found (Eq. (2.21)).

Before setting up the prong method, it might be useful to consider a few diagrams to illustrate approximants for three variables. Three variable approximants contain all the essential features of approximants for more than three variables. Figures 4.1–4.3 illustrate the definitions introduced in Section 2. Figure 4.1 represents a CA, Fig. 4.2 an SOD approximant, and Fig. 4.3 a GOD approximant. The points $\mathbf{p}$ that form the set P lie on the three surfaces OAD, OBD, and OCD. The prongs branch out from these points; prongs with three branches starting on the completely symmetric line OD. The various regions S_1, S_2, S_3 and S_4 are indicated on each diagram. S_2 is of course an empty set for a CA.

Figures 4.1–4.3 indicate that once the two variable approximants have been defined, and the rectangular rule is required for the three, or more, variable

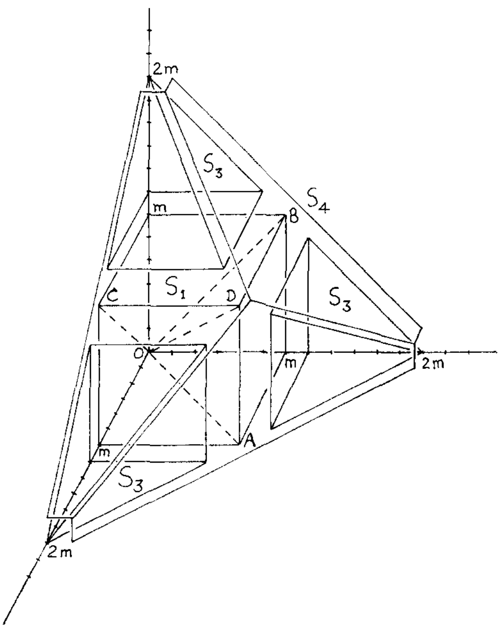

FIG. 4.1. CA for three variables.

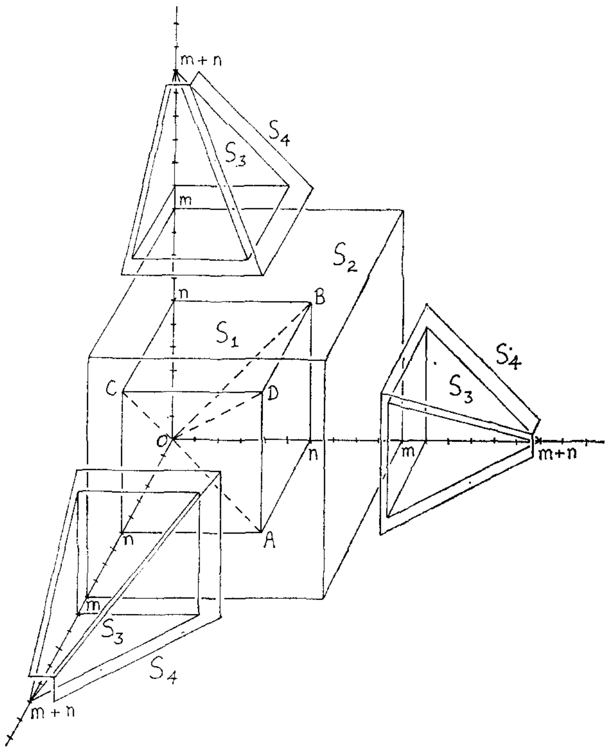

FIG. 4.2. SOD approximant for three variables.

approximants, there is really no choice in the definition of approximants in more than two variables. The regions S_1, S_2, S_3, and S_4 are defined by projecting upwards away from each coordinate plane.

The first step in constructing the prong method for N-variables is contained in the argument leading up to Theorem 2.1, namely, that the number of coefficients b_σ and a_μ attached to the prong $R_\mathbf{p}$ is equal to the number of defining equations attached to the same prong, i.e., to the number of elements in $E_\mathbf{p}$. As each a_μ is given by one equation in $E_\mathbf{p}$, it follows that the number of coefficients b_σ attached to $R_\mathbf{p}$ equals the number of equations attached to $R_\mathbf{p}$ that do not involve the a_μ. This result will be used in the following.

The second step in constructing the prong method lies in the introduction of a partial ordering for the points $\mathbf{p}$ in the set P. The prongs $R_\mathbf{p}$ then will be ordered according to this rule. In Section 2, the partial ordering

$$\boldsymbol{\beta} < \boldsymbol{\alpha}, \qquad \text{means} \quad \boldsymbol{\beta} \neq \boldsymbol{\alpha}; \quad \beta_i \leqslant \alpha_i\,, \quad i \in I_N$$

was introduced (Eq. (2.20)). It will be shown that an ordering of the sets $R_{\mathbf{p}}$ according to this rule will result in a systematic method for solving the defining equations. A particular set of equations $E_{\mathbf{p}}$ will thus only be considered after all the sets $E_{\mathbf{q}}$, $\mathbf{q} < \mathbf{p}$ have been considered. For example, for three variables the set P may be ordered in the following manner:

$$\begin{array}{llll}
(0,0,0), & (1,1,0), & (2,2,0),\ldots \\
 & (1,0,1), & (2,0,2),\ldots \\
 & (0,1,1), & (0,2,2),\ldots \\
(1,1,1), & (2,2,1),\ldots & \\
 & (2,1,2),\ldots & \\
 & (1,2,2),\ldots & \\
(2,2,2),\ldots & &
\end{array}$$

Alternative orderings will of course exist for three or more variables as the above ordering is only a partial ordering. For two variables, the elements of P may be ordered in only one way.

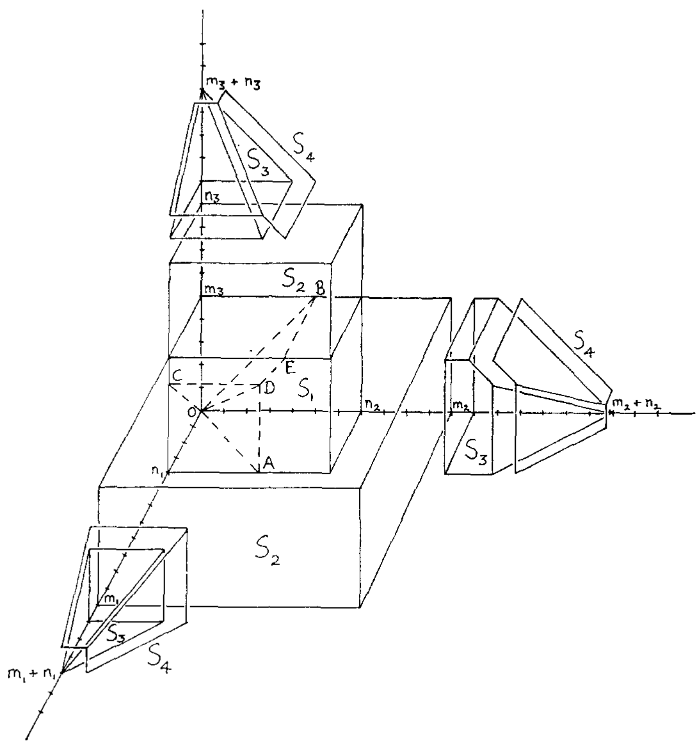

FIG. 4.3. GOD approximant for three variables.

The third stage in the construction of the prong method lies in the following theorem:

THEOREM 4.1. *The coefficients $b_{\boldsymbol{\sigma}}$ that occur in the equations that form the set $E_{\mathbf{p}}$ are attached to prongs $R_{\mathbf{q}}$, $\mathbf{q} \leqslant \mathbf{p}$.*

Proof. Let $\mathbf{e}_i$ be unit vectors in N dimensions given by

$$\begin{aligned} \mathbf{e}_1 &= (1, 0,\dots, 0) \\ \mathbf{e}_2 &= (0, 1,\dots, 0) \\ &\vdots \\ \mathbf{e}_N &= (0, 0,\dots, 1). \end{aligned} \tag{4.1}$$

This theorem will be proved by considering an example. Without any loss of generality, consider the prong $R_{\mathbf{p}}$ for which $\mathbf{p}$ is given by

$$\mathbf{p} = (p,\dots, p, p_{i+1},\dots, p_N); \qquad p_{i+1} < p,\dots, p_N < p.$$

Consider an equation $d_{\boldsymbol{\alpha}} = 0$ in the set $E_{\mathbf{p}}$. Without loss of generality, let

$$\boldsymbol{\alpha} = \mathbf{p} + (\alpha_1 - p)\,\mathbf{e}_1, \qquad \alpha_1 - p > 0,$$

i.e., $d_{\boldsymbol{\alpha}} = 0$ is an equation attached to a point on the first branch of $R_{\mathbf{p}}$. This equation is

$$\begin{aligned} \sum_{\boldsymbol{\sigma} \in S_{\mathrm{n}}} b_{\boldsymbol{\sigma}} c_{\boldsymbol{\alpha}-\boldsymbol{\sigma}} &= a_{\boldsymbol{\alpha}}, \qquad \boldsymbol{\alpha} \in S_{\mathrm{m}} \\ &= 0, \qquad \boldsymbol{\alpha} \notin S_{\mathrm{m}}, \end{aligned}$$

where we use the usual convention that $c_{\boldsymbol{\alpha}-\boldsymbol{\sigma}}$ is zero if one or more of the components of $\boldsymbol{\alpha} - \boldsymbol{\sigma}$ is negative. The vector index $\boldsymbol{\sigma}$ in the above equation thus lies in the range

$$\mathbf{0} \leqslant \boldsymbol{\sigma} \leqslant \boldsymbol{\sigma}_m,$$

where

$$\begin{aligned} \boldsymbol{\sigma}_m &= \boldsymbol{\alpha}, \qquad \text{if} \quad \alpha_1 \leqslant n_1, \\ &= \boldsymbol{\alpha} - (\alpha_1 - n_1)\,\mathbf{e}_1, \qquad \text{if} \quad \alpha_1 > n_1. \end{aligned}$$

If a particular value of $\boldsymbol{\sigma}$ that occurs in the above equation has components given by

$$\sigma_1 = \sigma_{m1} - \beta_1; \qquad \sigma_i = \sigma_{mi}, \quad i = 2,\dots, N,$$

then $b_{\boldsymbol{\sigma}}$ is attached to $R_{\mathbf{p}}$ if $\sigma_{m1} - \beta_1 \geqslant p$, and $b_{\boldsymbol{\sigma}}$ is attached to prong $R_{\mathbf{q}}$ for which $\mathbf{q} < \mathbf{p}$ if $\sigma_{m1} - \beta_1 < p$.

If another particular $\boldsymbol{\sigma}$ has components such that

$$\sigma_1 \leqslant \sigma_{m1}\,; \qquad \sigma_i < \sigma_{mi}\,, \text{ for at least one } i \text{ in the range } 2,..., N,$$

then it is straightforward to prove, by considering various cases, that $b_{\boldsymbol{\sigma}}$ is attached to a prong $R_{\mathbf{q}}$ for which $\mathbf{q} < \mathbf{p}$.

As the points $\mathbf{p}$ in the set P, and hence, the prongs $R_{\mathbf{p}}$, have been partially ordered so that $E_{\mathbf{p}}$ is considered only when all $E_{\mathbf{q}}$, $\mathbf{q} < \mathbf{p}$ have been considered, we have, using the result of this theorem, a systematic method, the prong method, for solving the complete set of defining equations for the approximant $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$.

Consider a particular prong $R_{\mathbf{p}}$. The set $I_{\mathbf{p}}$ is given by

$$I_{\mathbf{p}} = \{j \mid p_j = \max_{i \in I_N} p_i = p\}.$$

Let

$$I_{\mathbf{p}} = \{i_1 ,..., i_l\}. \tag{4.2}$$

$I_{\mathbf{p}}$ thus has l elements. The coefficients $b_{\boldsymbol{\sigma}}$ attached to $R_{\mathbf{p}}$ form a vector $\mathbf{b}_{\mathbf{p}}$ defined by

$$\mathbf{b}_{\mathbf{p}} = \begin{pmatrix} \mathbf{b}_{\mathbf{p};i_1} \\ \vdots \\ \mathbf{b}_{\mathbf{p};i_l} \\ b_{\mathbf{p}} \end{pmatrix}, \tag{4.3}$$

where

$$\mathbf{b}_{\mathbf{p};i} = \begin{pmatrix} b_{\mathbf{p}+(n_i-p)\mathbf{e}_i} \\ \vdots \\ b_{\mathbf{p}+\mathbf{e}_i} \end{pmatrix}, \qquad i \in I_{\mathbf{p}} \tag{4.4}$$

Let

$$C_{p;i} = \begin{pmatrix} c_{(m_i-n_i+1)\mathbf{e}_i} & \cdots & c_{(m_i-p)\mathbf{e}_i} \\ \vdots & & \vdots \\ c_{(m_i-p)\mathbf{e}_i} & \cdots & c_{(m_i+n_i-2p-1)\mathbf{e}_i} \end{pmatrix} \tag{4.5}$$

$$X_{p;i} = (c_{(m_i-p+1)\mathbf{e}_i} \quad \cdots \quad c_{(m_i+n_i-2p)\mathbf{e}_i})^T \tag{4.6}$$

$$Y_{p;i} = c_{(m_i+n_i-2p+1)\mathbf{e}_i}\,. \tag{4.7}$$

If all the $b_{\boldsymbol{\sigma}}$ attached to the prongs $R_{\mathbf{q}}$, $\mathbf{q} < \mathbf{p}$, have already been found, the equations in the set $E_{\mathbf{p}}$ that determine $\mathbf{b}_{\mathbf{p}}$ are of the form

$$D_{\mathbf{p}}\mathbf{b}_{\mathbf{p}} = \text{quantity involving known coefficients,} \tag{4.8}$$

where

$$D_{\mathbf{p}} = \begin{pmatrix} C_{p;i_1} & & 0 & X_{p;i_1} \\ & \ddots & & \vdots \\ 0 & & C_{p;i_l} & X_{p;i_l} \\ X^T_{p;i_1} & \cdots & X^T_{p;i_l} & \sum_{i \in I_{\mathbf{p}}} Y_{p;i} \end{pmatrix}. \tag{4.9}$$

The matrix D_0 is slightly different from this general form since $R_{4;0} = \varnothing$ (the empty set), and we normalize the approximant by taking $b_0 = 1$. D_0 is thus given by

$$D_0 = \begin{pmatrix} C_{0;1} & & 0 & X_{0;1} \\ & \ddots & & \vdots \\ & & C_{0;N} & X_{0;N} \\ 0 & & & 1 \end{pmatrix}, \tag{4.10}$$

and $\mathbf{b}_0$ is determined by an equation $D_0\mathbf{b}_0$ = column vector with unity in last position and zeros elsewhere.

Thus, when the prong method is carried through for all $\mathbf{p} \in P$, we obtain for each $\mathbf{p}$ a matrix $D_{\mathbf{p}}$ that has to be nonsingular if the approximant $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ is to be nondegenerate. D_0 is of course nondegenerate if and only if

$$C_{0;i}, \qquad i = 1,\ldots, N \text{ are all nonsingular matrices.}$$

It may be noted that not all matrices $D_{\mathbf{p}}$ are necessarily different for series in more than two variables. For example, for three variables, there are equalities of the form

$$D_{\mathbf{p}_1} = D_{\mathbf{p}_2}, \qquad \text{if} \quad \mathbf{p}_1 = (p, p, p_3), \quad \mathbf{p}_2 = (p, p, p_3 - 1), \quad p_3 < p.$$

All coefficients b_σ (and hence, a_μ) attached to points in S_1 and $\bigcup_{\mathbf{p}\in P} R_{2;\mathbf{p}}$ are determined during the above systematic method. In general, there are also coefficients attached to

$$S_5 = S_2 \Big\backslash \bigcup_{\mathbf{p}\in P} R_{2;\mathbf{p}},$$

which remain to be determined. Again, as for the case of two variables, if $c_0 \neq 0$, any remaining b_σ may be determined one at a time by imposing the partial ordering on the points in S_5. Any coefficients a_μ attached to S_5 then may be determined immediately.

The complete set of equations for all the b_σ in $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ may be written in a lower triangular block form similar to Eq. (3.15). The following theorem has now been proved.

THEOREM 4.2. *$f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ is a nondegenerate approximant if and only if*

(i) *$D_{\mathbf{p}}$, $\mathbf{p} \in P$, are all nonsingular matrices,*

(ii) *$c_0 \neq 0$, if $S_{\mathbf{n}} \cap S_5 \neq \varnothing$.*

As for the case of two variables, it may be seen that the conditions for nondegeneracy involve only the coefficients in the one variable series obtained from $f(\mathbf{z})$ by equating each of the other variables to zero, i.e., the series

$$g^i(z_i) = \sum_{\alpha=0}^{\infty} c_{\alpha \mathbf{e}_i} z_i^{\alpha}, \qquad i = 1,..., N. \tag{4.11}$$

The partial ordering introduced in the construction of the prong method implies a "projection" theorem:

THEOREM 4.3. *Let $f(\mathbf{z})$ be a power series in N variables and let $\hat{f}(\hat{\mathbf{z}})$ be a power series in $\hat{N}$ variables ($\hat{N} < N$) obtained from $f(\mathbf{z})$ by equating to zero $N - \hat{N}$ of the variables $z_1,..., z_N$. Write*

$$I_{\hat{N}} = \{i_1,..., i_{\hat{N}}\} \subset I_N,$$

$$\hat{\mathbf{m}} = (m_{i_1},..., m_{i_{\hat{N}}}),$$

$$\hat{\mathbf{n}} = (n_{i_1},..., n_{i_{\hat{N}}}).$$

Then, $\hat{f}_{\hat{\mathbf{m}}/\hat{\mathbf{n}}}(\hat{\mathbf{z}}) = f_{\mathbf{m}/\mathbf{n}}(\mathbf{z}) \,|_{z_i=0;\, i \in I_N \setminus I_{\hat{N}}}$.

Proof. The proof of this theorem lies in the fact that all prongs $R_{\mathbf{p}}$ with a particular component, or components, of $\mathbf{p}$ zero may be considered before prongs $R_{\mathbf{p}}$ with that particular component, or components, of $\mathbf{p}$ nonzero. Note also that the set of equations $D_0 \mathbf{b}_0$ = (column vector with unity in last position and zeros elsewhere) may be split into N independent sets of equations

$$C_{0;i} \mathbf{b}_{0;i} = -X_{0;i}, \qquad i = 1,..., N. \tag{4.12}$$

This theorem includes as a particular case the reduction to Padé approximants when all but one of the variables is equated to zero.

When the prong method is used numerically to solve for the coefficients b_σ, a method of triangulating each of the matrices $D_{\mathbf{p}}$, $\mathbf{p} \in P$, is required. One method for carrying out this procedure is a generalization of the method given in the previous section for two variables. Assume, for the moment, that the series $g^i(z_i)$, $i = 1,..., N$, are all normal series. The matrices $C_{p;i}$ will then all be nonsingular. Introduce, as in Eqs. (3.17)–(3.19), matrices $\tilde{C}_{p;i}$, $\tilde{D}_{\mathbf{p}}$ and vectors $\tilde{X}_{p;i}$ such that all leading minors of the $\tilde{C}_{p;i}$ are nonsingular. Unit lower triangular matrices $L_{p;i}$ then may be found such that

$L_{p;i}\tilde{C}_{p;i}$ are upper triangular matrices. As for the case of two variables, there are recurrence relations of the form

$$\tilde{C}_{p-1;i} = \begin{pmatrix} \tilde{C}_{p;i} & \tilde{X}_{p;i} \\ X^T_{p;i} & Y_{p;i} \end{pmatrix}, \qquad i = 1,\ldots, N, \tag{4.13}$$

and

$$L_{p-1;i} = \begin{pmatrix} L_{p;i} & 0 \\ W_{p;i} & 1 \end{pmatrix}, \qquad i = 1,\ldots, N. \tag{4.14}$$

The matrix $\tilde{D}_{\mathbf{p}}$, for a particular value of $\mathbf{p}$, will be reduced to upper triangular form by

$$L_{\mathbf{p}} = \begin{pmatrix} L_{p;i_1} & & & 0 & \\ & \ddots & & & \\ 0 & & & L_{p;i_l} & \\ W_{p;i_1} & \cdots & & W_{p;i_l} & 1 \end{pmatrix}, \tag{4.15}$$

where

$$I_{\mathbf{p}} = \{i_1, \ldots, i_l\} = \{j \mid p_j = \max_{i \in I_N} p_i = p\} \text{ as usual.}$$

L_0 will have zeros instead of W's in the last row. Once again, the lowest order lower triangular matrices, namely,

$$L_{0;i}, \qquad i = 1, 2, \ldots, N,$$

contain all the information required to write down all lower triangular matrices $L_{\mathbf{p}}$, $\mathbf{p} \in P$. Thus, when calculating $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$, the amount of elimination necessary equals the amount of elimination necessary in calculating the Padé approximants

$$g^i_{m_i/n_i}(z_i), \qquad i = 1,\ldots, N.$$

The various theorems proved in the previous section now may be extended to the N-variable case.

THEOREM 4.4. *The approximant $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ is nondegenerate if*

$$\begin{aligned} &\text{(i)} \quad g^i(z_i),\ i = 1,\ldots, N, \text{ are all normal series,} \\ &\text{(ii)} \quad \Omega_{\mathbf{p}} = \sum_{i \in I_{\mathbf{p}}} (W_{p;i}\tilde{X}_{p;i} + Y_{p;i}) \neq 0,\ \mathbf{p} \in P \setminus \{\mathbf{0}\}. \end{aligned} \tag{4.16}$$

The proof of this theorem is similar to the proof of Theorem 3.3.

THEOREM 4.5. *The approximant $\hat{f}_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ to the series*

$$\hat{f}(\mathbf{z}) \equiv f(K^{(1)}z_1, \ldots, K^{(N)}z_N)$$

will be nondegenerate if

(i) $g^i(z_i)$, $i = 1, \ldots, N$ *are normal series,*

(ii) *the scale factors $K^{(i)}$ do not satisfy a finite number of polynomial equations.*

Proof. This theorem may be proved in a similar fashion to the method of proving Theorem 3.4. It may be shown that the $K^{(i)}$ must be such that

$$\sum_{i \in I_{\mathbf{p}}} K_p^{(i)}(W_{p;i}\tilde{X}_{p;i} + Y_{p;i}) \neq 0, \qquad \mathbf{p} \in P\backslash\{\mathbf{0}\}, \tag{4.17}$$

where

$$K_p^{(i)} = (K^{(i)})^{m_i+n_i-2p+1}. \tag{4.18}$$

THEOREM 4.6. *The series $\hat{f}(\mathbf{z})$ (as defined in Theorem 4.5) is a normal series, i.e., approximants $\hat{f}_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$, for all $\mathbf{m}$ and $\mathbf{n}$, are nondegenerate, if and only if*

(i) $g^i(z_i)$, $i = 1, \ldots, N$, *are normal series,*

(ii) *the scale factors $K^{(i)}$ do not satisfy a denumerable set of polynomial equations.*

Proof. The first condition is needed as $\hat{f}_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ reduces to a Padé approximant if all but one of the variables is equated to zero. The second condition may be seen to be needed by letting $\mathbf{m}$ and $\mathbf{n}$ take on all possible values in the previous theorem.

If $f(\mathbf{z})$ is a function symmetric in all, or some, of its variables, then Theorems 3.6–3.11 have the obvious generalizations.

Theorems 3.12 and 3.13, concerning factorization and the inverse function, may also be generalized to N variables.

THEOREM 4.7. *Let $f(\mathbf{z})$ be a function of N variables that factorizes into the product of two functions of N_1 variables and N_2 variables ($N = N_1 + N_2$). Without loss of generality, write*

$$f(\mathbf{z}) = f^1(\mathbf{z}^1) f^2(\mathbf{z}^2) = f^1(z_1, \ldots, z_{N_1}) f^2(z_{N_1+1}, \ldots, z_N). \tag{4.19}$$

Then, assuming the approximants are nondegenerate,

$$f_{\mathbf{m}/\mathbf{n}}(\mathbf{z}) = f^1_{\mathbf{m}^1/\mathbf{n}^1}(\mathbf{z}^1) f^2_{\mathbf{m}^2/\mathbf{n}^2}(\mathbf{z}^2), \tag{4.20}$$

where

$$\mathbf{m}^1 = (m_1, \ldots, m_{N_1}), \qquad \mathbf{m}^2 = (m_{N_1+1}, \ldots, m_N),$$
$$\mathbf{n}^1 = (n_1, \ldots, n_{N_1}), \qquad \mathbf{n}^2 = (n_{N_1+1}, \ldots, n_N).$$

Proof. Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}^1 \oplus \boldsymbol{\alpha}^2$ signify the splitting of a vector with N components into two vectors, with N_1 and N_2 components, e.g.,

$$\mathbf{z} = \mathbf{z}^1 \oplus \mathbf{z}^2, \qquad \mathbf{m} = \mathbf{m}^1 \oplus \mathbf{m}^2, \qquad \mathbf{n} = \mathbf{n}^1 \oplus \mathbf{n}^2.$$

The approximants $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$, $f^1_{\mathbf{m}^1/\mathbf{n}^1}(\mathbf{z}^1)$ and $f^2_{\mathbf{m}^2/\mathbf{n}^2}(\mathbf{z}^2)$ are defined by

$$\sum_{\boldsymbol{\sigma}\in S_\mathbf{n}} b_{\boldsymbol{\sigma}} \mathbf{z}^{\boldsymbol{\sigma}} \sum_{\boldsymbol{\alpha}\in S} c_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\mu}\in S_\mathbf{m}} a_{\boldsymbol{\mu}} \mathbf{z}^{\boldsymbol{\mu}} + \sum_{\boldsymbol{\beta}\in S} d_{\boldsymbol{\beta}} \mathbf{z}^{\boldsymbol{\beta}}$$

$$\sum_{\boldsymbol{\sigma}^1\in S^1_{\mathbf{n}^1}} b^1_{\boldsymbol{\sigma}^1} (\mathbf{z}^1)^{\boldsymbol{\sigma}^1} \sum_{\boldsymbol{\alpha}^1\in S^1} c^1_{\boldsymbol{\alpha}^1} (\mathbf{z}^1)^{\boldsymbol{\alpha}^1} = \sum_{\boldsymbol{\mu}^1\in S^1_{\mathbf{m}^1}} a^1_{\boldsymbol{\mu}^1} (\mathbf{z}^1)^{\boldsymbol{\mu}^1} + \sum_{\boldsymbol{\beta}^1\in S^1} d^1_{\boldsymbol{\beta}^1} (\mathbf{z}^1)^{\boldsymbol{\beta}^1}$$

$$\sum_{\boldsymbol{\sigma}^2\in S^2_{\mathbf{n}^2}} b^2_{\boldsymbol{\sigma}^2} (\mathbf{z}^2)^{\boldsymbol{\sigma}^2} \sum_{\boldsymbol{\alpha}^2\in S^2} c^2_{\boldsymbol{\alpha}^2} (\mathbf{z}^2)^{\boldsymbol{\alpha}^2} = \sum_{\boldsymbol{\mu}^2\in S^2_{\mathbf{m}^2}} a^2_{\boldsymbol{\mu}^2} (\mathbf{z}^2)^{\boldsymbol{\mu}^2} + \sum_{\boldsymbol{\beta}^2\in S^2} d^2_{\boldsymbol{\beta}^2} (\mathbf{z}^2)^{\boldsymbol{\beta}^2},$$

where

$$d_{\boldsymbol{\beta}} = 0, \qquad \boldsymbol{\beta} \in S_1 \cup S_2 \cup S_3$$
$$d^1_{\boldsymbol{\beta}^1} = 0, \qquad \boldsymbol{\beta}^1 \in S_1{}^1 \cup S_2{}^1 \cup S_3{}^1$$
$$d^2_{\boldsymbol{\beta}^2} = 0, \qquad \boldsymbol{\beta}^2 \in S_1{}^2 \cup S_2{}^2 \cup S_3{}^2,$$

and the sums of various d's are zero over the sets S_4, $S_4{}^1$, $S_4{}^2$.

The following properties of the various S's may be proved from their definitions (reference may be made to Figs. 4.1–4.3 for a demonstration of these properties for 2 + 1 variables):

$$S = S^1 \oplus S^2$$
$$S_\mathbf{m} = S^1_{\mathbf{m}^1} \oplus S^2_{\mathbf{m}^2}$$
$$S_\mathbf{n} = S^1_{\mathbf{n}^1} \oplus S^2_{\mathbf{n}^2}$$
$$(S_1 \cup S_2 \cup S_3) \subset (S_1{}^1 \cup S_2{}^1 \cup S_3{}^1) \oplus (S_1{}^2 \cup S_2{}^2 \cup S_3{}^2)$$
$$(S_1 \cup S_2 \cup S_3 \cup S_4) \subset (S_1{}^1 \cup S_2{}^1 \cup S_3{}^1 \cup S_4{}^1) \oplus (S_1{}^2 \cup S_2{}^2 \cup S_3{}^2 \cup S_4{}^2).$$

These properties ensure that, on multiplying the defining equations for $f^1_{\mathbf{m}^1/\mathbf{n}^1}(\mathbf{z}^1)$ and $f^2_{\mathbf{m}^2/\mathbf{n}^2}(\mathbf{z}^2)$, we obtain the defining equation for $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ (with an extra bonus in having some $d_{\boldsymbol{\beta}} = 0$ outside the region $S_1 \cup S_2 \cup S_3$). The theorem is thus proved. The theorem of course may be applied several times to a function that factorizes into the product of more than two functions.

THEOREM 4.8. *Let $f(\mathbf{z})$ be a series with an inverse series $g(\mathbf{z})$ in the sense that*

$$f(\mathbf{z})\, g(\mathbf{z}) = 1. \tag{4.21}$$

If $f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})$ and $g_{\mathbf{n}/\mathbf{m}}(\mathbf{z})$ are nondegenerate, then

$$f_{\mathbf{m}/\mathbf{n}}(\mathbf{z})\, g_{\mathbf{n}/\mathbf{m}}(\mathbf{z}) = 1. \tag{4.22}$$

Proof. This theorem may be proved in the same way as Theorem 3.13. One notes that S_1, S_2, S_3, and S_4 are invariant under the interchange of $\mathbf{m}$ and $\mathbf{n}$. Also, the sets $S_1 \cup S_2 \cup S_3$ and $S_1 \cup S_2 \cup S_3 \cup S_4$ obey the rectangular rule.

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